Name: Date: Class:

Tower Height, Surface Area and Volume Quiz

Answer Key

Assume we are using six sheets of A3 (279 mm x 420 mm) cardstock to build our tower. Recall from our investigation of the tower that we found that the volumes and surface areas of successive blocks to be in geometric progressions.

1. Fill in the common ratios

   a. Common ratio for block volumes: $r_v = \frac{\sqrt{2}}{4}$

   b. Common ratio for block surface areas: $r_a = \frac{1}{2}$

2. Now consider the block heights. Notice that these are also in a geometric progression. Find the common ratio:

   $r_h = \frac{\sqrt{2}}{2}$

3. It turns out the geometric progressions can be summed up when the common ratio is between zero and one. This problem will guide you through the process of understanding this concept.

   a. Let v be the volume of the first block. $v = 27.9 \text{ cm} \times 21 \text{ cm} \times 21 \text{ cm} \approx 12{,}304 \text{ cc}$

   b. Given the volume v of the first block and the common ratio r_v we can find the volume of the entire tower V_T like this:

$$V_T = v + v \cdot (r_v) + v \cdot (r_v)^2 + v \cdot (r_v)^3 + \cdots$$
$$= v \cdot (1 + r_v + r_v^2 + r_v^3 + \cdots)$$

The trick is to find the value of the sum of the powers of the common ratio r_v. Here's the trick. Let $1 + r_v + r_v^2 + r_v^3 + \cdots = s$. We can subtract 1 from both sides to get:

$$r_v + r_v^2 + r_v^3 + \cdots = s - 1$$

Then we factor out the common ratio on the left-hand side:

$$r_v \cdot (1 + r_v + r_v^2 + r_v^3 + \cdots) = s - 1$$

Notice that the sum in the parentheses is the original sum s!

Substitute the s in to the equation and solve for it.

$$r_v \cdot s = s - 1 \Rightarrow s = \frac{1}{1 - r_v} = \frac{1}{1 - \sqrt{2}/4} = \frac{2}{7}(4 + \sqrt{2})$$

$s =$ ______________

c. Now we can find the volume of the tower!

$$V_T = v \cdot s = 12{,}304 \text{ cc} \cdot \frac{2}{7}(4 + \sqrt{2}) \approx 19{,}033 \text{ cc}$$

$V_T =$ ______________

4. a. Repeat the process in problem (3) to find the surface area A_T of the whole tower. Recall that the bottom of the blocks are not part of the surface area nor is the section of the top that the subsequent block rests on!

Surface area of first block:

$$a = 2 \times 21 \text{ cm} \times 21 \text{ cm} + 3 \times 27.9 \text{ cm} \times 21 \text{ cm} - 21 \text{ cm} \times \frac{27.9}{2} \text{ cm}$$
$$= 2346.75 \text{ sq cm}$$

Surface area of entire tower:

$$A_T = a \cdot \frac{1}{1 - 1/2} = 4693.5 \text{ sq cm}$$

This implies that the same amount of cardstock used to make the first box is the same amount that would need to be used for all the others.

b. In the project we learned where to position the elevator shaft so that we could reach any floor. Now determine how tall the elevator shaft would need to be to reach them.

The height of the first block is 21 cm so $H_T = 21 \text{ cm} \cdot \frac{1}{1-\sqrt{2}/2} \approx 71.7 \text{ cm}$

5. The tallest building in the world (as of May, 2020) is the Burj Khalifa in Dubai, UAE topping out at an impressive 2,722 ft which is over half a mile high! You're hired by a developer to design a tower like we made with cardstock that is to be 3,000 ft high, so it will replace the Burj Khalifa as the tallest building in the world.

a. Realistically the elevator shaft will only go as high as the smallest inhabitable room, the penthouse. We define inhabitable to mean having a ceiling that is at least 8 ft high. Which floor will the elevator shaft reach under these conditions?

The first story being about 878.68 feet high ($3000 \cdot \left(1 - \frac{\sqrt{2}}{2}\right)$ exactly). Since the height of the n^{th} story is product of this height and $n - 1$ factors of r_h we have:

$$8 = 3000 \cdot \left(1 - \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{2}}{2}\right)^{n-1} \Rightarrow n \approx 14.56$$

So, the penthouse is the 14th story of the building giving a ceiling height of about 9.7 feet. If there were a 15th story it would only have a ceiling height of about 6.9 feet.

b. Assuming the surface will be mostly glass and that this glass costs about $100 per square foot, how much will all the glass cost to make the tower?

The surface area of the main block (excluding the bottom and part of the top) can be easily calculated in the same way as in 4(a) and when doubled is about 4819623 sq ft. This means the glass cost is roughly $482 million.

c. How much real estate does this tower require? In other words, what is the building footprint at the ground level in square feet?

$\left[3000\left(1 - \frac{\sqrt{2}}{2}\right)\right]^2 \cdot \sqrt{2} \approx 1091883$ sq ft ≈ 25 acres which is close to the Pentagon