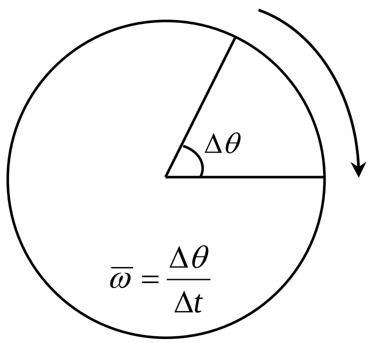
Name: ____ Date: ___________________ Class: ________________________

A Tale of Friction Student Notes

1.0 Basic Concepts

1.1 Rotational Movement Kinematics

Angular Velocity Definition

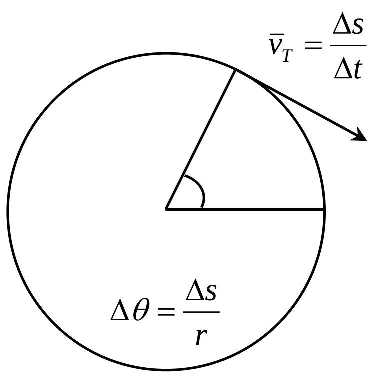


Average angular velocity: $\overline{\omega} = \frac{\Delta\theta}{\Delta t}$

Instantaneous angular velocity: $\omega = \frac{d\theta}{dt}$

angle θ: radians

Tangential Velocity



Average tangential velocity: $\bar{v}_T = \frac{\Delta s}{\Delta t}$

Instantaneous tangential velocity: $v_T = \frac{ds}{dt}$

s: length of arc

Because $\theta = \frac{s}{r}$,(r = circle radius, s = arc length), $s = \theta \cdot r$,then the angular and tangential velocities are related as:

$$v_T = r\,\frac{d\theta}{dt} \quad \text{or} \quad v_T = r \cdot \omega$$

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If angular acceleration is defined as: $\alpha = \frac{d\omega}{dt}$, and tangential acceleration as: $a_T = \frac{dv_T}{dt}$, then

$$\alpha = \frac{d^2\theta}{dt^2} \quad \text{and} \quad a_T = r\frac{d^2\theta}{dt^2}$$

The angular and tangential accelerations are then related as: $a_{T.} = \alpha \cdot r$

1.2 Rotational Kinetic Energy and Moment of Inertia of a Rigid Body

For a single particle, the kinetic energy for linear movement is defined as: $K = ½\, mv^2$. Similarly, for a single particle, the kinetic energy for rotational movement is given by the formula $K = ½\, I\omega^2$, where $I = m\, r^2$ is known as moment of inertia of the particle, and r is the distance from the point of rotation.

For a system of particles, its moment of inertia is the sum of the individual moments $I = \Sigma m_i r_i^2$. This definition can be extended to compute the moment of inertia for a continuous rigid body:

$$I = \int r^2 dm$$

For the special case analyzed in this project, the moment of inertia for a solid homogeneous sphere is:

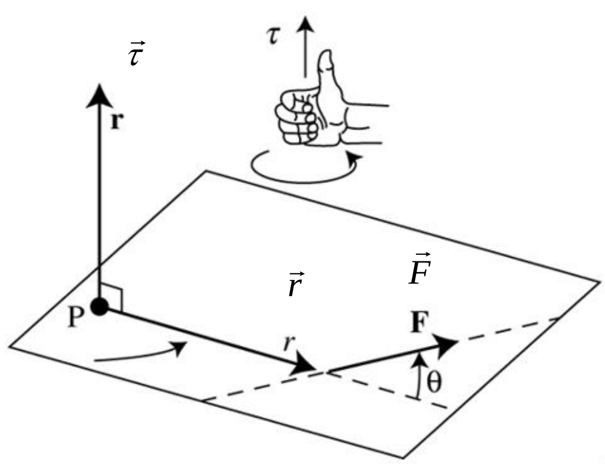
$$I = \frac{2}{5} r^2, \qquad r = \text{radius of the sphere}$$

1.3 Angular Momentum and Torque of a Rigid Body

From the law of lever, the torque is defined as $\tau = F \cdot d$, where F is the perpendicular force applied on the lever, at a distance d from the fulcrum.

Extending this definition, taking the force F as a vector $\vec{F}$ non- perpendicular to the vector $\vec{r}$ that gives the position of the point where the force is applied with respect to the rotation point, the torque is defin $\vec{\tau}$ as the vector product:

$$\vec{\tau} = \vec{r} \times \vec{F}$$



Where the torque $\vec{\tau}$ is represented as a vector perpendicular to the plane defined by vectors $\vec{r}$ and $\vec{F}$.

The magnitude of this vector is given by the product of the magnitudes of the vectors $\vec{r}$ and $\vec{F}$, (r and F) times the sine of the angle between them:

$$\tau = r \cdot F \cdot \sin(\theta)$$

When the angle is 90°, $\vec{r}$ and $\vec{F}$ are perpendicular, the original law of lever is obtained.

For a single particle, the linear momentum is defined as the vector: $\vec{P} = m \cdot \vec{v}$, and by definition, the force or Newton's second law for a single particle when its mass changes with the time is given by:

$$\vec{F} = \frac{d}{dt}(m \cdot \vec{v})$$

When the mass of the particle is constant:

$$\vec{F} = m\frac{d\vec{v}}{dt} = m \cdot \vec{a}$$

Similar to the linear momentum, the angular momentum of a particle is defined as:

$$\vec{L} = \vec{r} \times \vec{p} = m \cdot (\vec{r} \times \vec{v})$$

and similar to the force definition for linear movement in terms of the linear momentum, the force producing a rotational movement, or torque, is defined as:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

For a particle of constant mass in a circular movement (*r* constant), and *v* the tangential velocity ($\theta = 90^o$), the magnitude of the torque is given by:

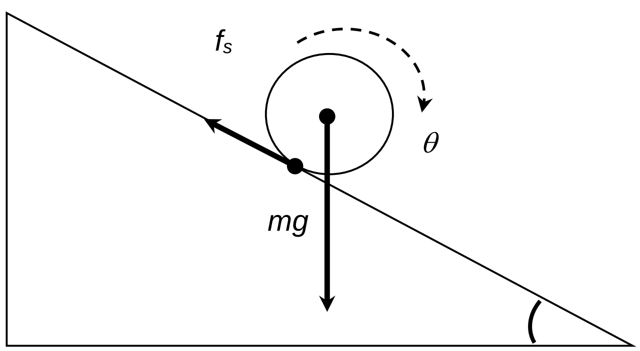
$$\tau = m \cdot r\frac{dv_T}{dt} = m \cdot r \cdot a_T = m \cdot r^2 \cdot \alpha$$

or

$$\tau = I \cdot \alpha$$

a formula similar to Newton's second law, for a rotating rigid body with constant mass.

2. Force of Friction for a Spherical Rigid Body Rolling without Slipping on an Incline Surface

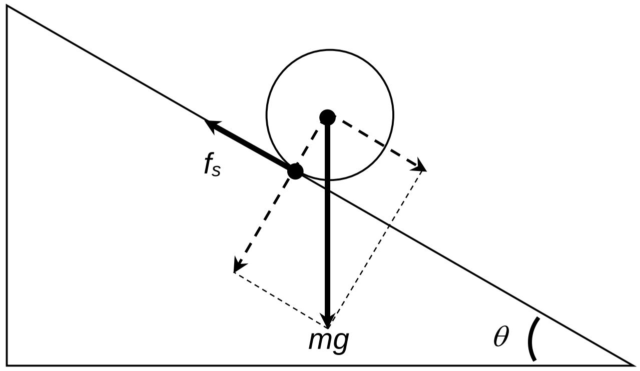


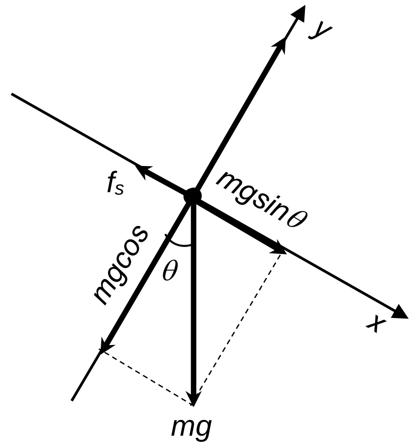
A spherical rigid body of mass *m* is rolling down an incline surface. The forces producing this movement are the weight of the body *mg* and the force of friction f_s.

Because the body is rolling instead sliding, the friction to consider in this problem is static friction. So, the coefficient to use in the friction force calculations is the static friction coefficient μ_{sl}.

The body rolls because a torque is produced by the friction force f_s and the component of the body's weight parallel to the inclined surface.

All forces on the rolling body can be analyzed by using a free-body diagram:





The forces that make this spherical solid of radius *r* and mass *m* roll down the incline plane are those along the *x-axis*. Using Newton's second law to describe the solids linear movement:

(1) $$F = m \cdot a = m \cdot g \cdot \sin\theta - f_s$$

The force that makes the solid rotate is the torque produced by the friction force:

(2) $$\tau = f_s \cdot r = I \cdot \alpha$$

where *I* = sphere's momentum of inertia, *α* = sphere's angular acceleration, *r* = sphere's radius.

For a homogenous spherical object: $I = \frac{2}{5} m \cdot r^2$. Substituting this value, and *α* = *a*/*r* in (2):

$$f_s \cdot r = \frac{2}{5} m \cdot r^2 \cdot \frac{a}{r}$$

The force of friction can be expressed as:

(3) $$f_s = \frac{2}{5} m \cdot a$$

Substituting (3) in (1):

$$m \cdot a = m \cdot g \cdot \sin\theta - \frac{2}{5} m \cdot a$$

$$a = g \cdot \sin\theta - \frac{2}{5} a$$

$$a + \frac{2}{5} a = g \cdot \sin\theta$$

$$\frac{7}{5} a = g \cdot \sin\theta$$

$$a = \frac{5}{7} g \cdot \sin\theta$$

Substituting the obtained acceleration value in (3), the force of friction can be expressed in terms of the sphere's weight and the angle of the incline:

(4)
$$f_s = \frac{2}{5} m \cdot \frac{5}{7} g \cdot \sin\theta$$
$$f_s = \frac{2}{7} m \cdot g \sin\theta$$

By definition, the static friction force is proportional to the normal force, or force the surface applies on the sphere to balance the component of the sphere's weight perpendicular to the surface.

(5)
$$f_s = \mu_s \cdot F_n$$

where μ_s is the static friction coefficient and F_n is the magnitude of the normal force. For this problem, the value of the normal force can be found in the free-body diagram.

$$F_n = m \cdot g \cdot \cos\theta$$

Substituting this value in (5)

(6)
$$f_s = \mu_s \cdot m \cdot g \cdot \cos\theta$$

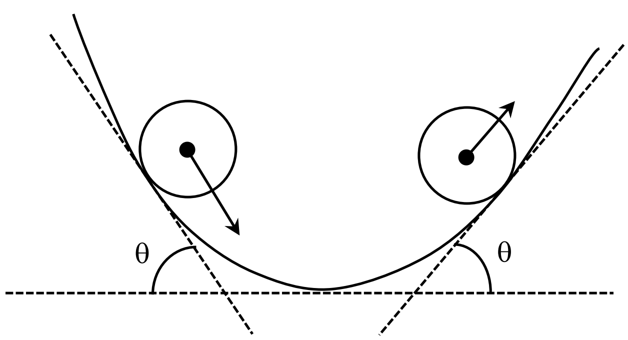
Combining equations (4) and (6), an expression for the coefficient of static friction can be found:

$$\mu_s \cdot m \cdot g \cdot \cos\theta = \frac{2}{7} m \cdot g \cdot \sin\theta$$
$$\mu_s \cdot \cos\theta = \frac{2}{7} \sin\theta$$
$$\mu_s = \frac{2}{7} \frac{\sin\theta}{\cos\theta}$$

(7)
$$\mu_s = \frac{2}{7} \tan\theta$$

Equation (7) states that for a spherical body rolling on an incline, the coefficient of static friction is a function of the inclined surface angle, specifically, of the tangent of this angle or slope of the inclined surface.

3. Force of Friction for a Spherical Rigid Body Rolling without Slipping on a Variable Slope Path



A sphere rolling on a path with variable slope can be visualized as rolling on a sequence of inclined tangent straight surfaces.

If the path is given as a function $y = f(x)$, differentiable at every point, the slope of any these inclines, defined as $m = \tan\theta$, can be found using the derivative $f'(x)$:

$$m = \frac{dy}{dx} = f'(x)$$

Then:

$$m = \tan\theta = f'(x)$$

Using the above expression in (7), the coefficient of static friction for a sphere rolling on a variable slope path is given by:

(8

$$\mu_s = \frac{2}{7} f'(x)$$

Expression (8) states that the static friction coefficient varies from point to point on the path. Using equation (8) in (6), the static friction force for a rolling sphere of mass *m* is:

$$fs = \frac{2}{7} f'(x) \cdot m \cdot g \cdot \cos\theta$$

Using trigonometry, it is possible to express cos θ also in terms of $f'(x)$. Because tan $\theta = f'(x)$, then:

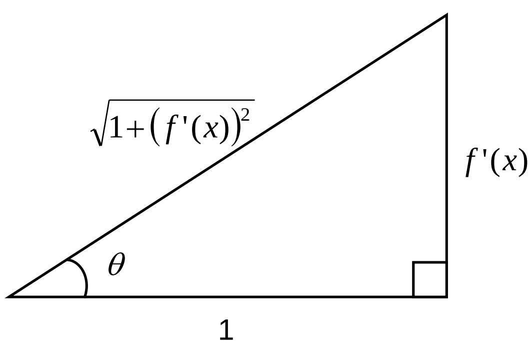
$$\theta = \arctan\left(f'(x)\right)$$

Then:

(9)

$$f_s = \frac{2}{7} f'(x) \cdot m \cdot g \cdot \cos(\arctan(f'(x)))$$

Solving with right triangles:



$$\tan\theta = \frac{f'(x)}{1} = \frac{opp}{adj}$$

$$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sqrt{1+\left(f'(x)\right)^2}}$$

$$\sin\theta = \frac{opp}{hyp} = \frac{f'(x)}{\sqrt{1+\left(f'(x)\right)^2}}$$

Using this expression for cos θ in (9):

(10)

$$f_s = \frac{2}{7} m \cdot g \cdot \frac{f'(x)}{\sqrt{1+\left(f'(x)\right)^2}}$$

Equations (8) and (10) seem to be adequate to model the problem of a sphere rolling on a variable slope path with friction, but something needs to be fixed. Both equations give negative values for some sections of the path, and by definition, the static friction coefficient is always positive.

The problem comes from the sign of the derivative and the possibility of negative slopes. For the inclined plane, the elevation angle is always positive. Because in this problem we approached the path with variable slope as a sequence of inclines, the slope given by $f'(x)$ must always be positive. With this restriction, equations (8) and (10) need to be given in terms of the absolute value of $f'(x)$:

(11)

$$\mu_s = \frac{2}{7}\left|f'(x)\right|$$

(12)

$$f_s = \frac{2}{7} m \cdot g \cdot \frac{|f'(x)|}{\sqrt{1+\left(f'(x)\right)^2}}$$

4. Work-Energy Theorem for a Sphere Rolling on a Variable Slope Path with Friction

The work-energy theorem states that the mechanical energy (kinetic energy + potential energy) of an isolated system under only conservative forces remains constant.

$$E_f = K_f + U_f = K_i + U_i = E_i$$

$$or$$

$$\Delta E = \Delta K + \Delta U = 0$$

Where *E* is the mechanical energy of the system, *K* is the kinetic energy ½ mv^2, *U* is the potential energy *mgh*, and the indexes $_f$ and $_i$ indicate the energies at the end and beginning of the process, respectively.

Energy cannot be created nor destroyed in an *isolated system*, but it can be internally converted to any other form of energy.

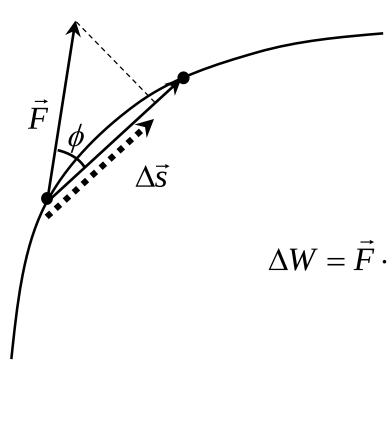
When a non-conservative force such as friction is considered in a system, the work-energy theorem states that the work done by the non-conservative forces, ΔW_f, is equivalent to the change in mechanical energy:

(13)

$$\Delta W_f = \Delta E = \Delta K + \Delta U$$

Under non-conservative forces, ΔE is no longer zero, but another key difference exists. Meanwhile for conservative systems, the work done by conservative forces depends only on the initial and final positions. For non-conservative systems, the work done by non-conservative forces, like friction, depends on the path or trajectory or on the time these forces affect the system. In a system under conservative forces, the work on a closed loop is zero, while in a system under non-conservative forces it is not.

By definition, mechanical work is the product of the displacement times the force component along the displacement:



$$\Delta W = \vec{F} \cdot \Delta\vec{s} = F \cdot \Delta s \cdot \cos\phi$$

For the variable slope path $y = f(x)$, the work done by the force of friction (12) is:

$$\Delta W = \frac{2}{7} m \cdot g \frac{|f'(x)|}{\sqrt{1 + \left(f'(x)\right)^2}} \cdot \Delta s$$

Taking differential displacements along the path:

(14) $$dW = \frac{2}{7} m \cdot g \frac{|f'(x)|}{\sqrt{1 + \left(f'(x)\right)^2}} \cdot ds$$

If the differential arc *ds* is defined as:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

the arc *ds* can be set as a function of $f'(x)$:

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)(dx)^2} \\ (15) \qquad &= \sqrt{1 + \left(f'(x)\right)^2} \cdot dx \end{aligned}$$

Substituting in (15) in (14):

$$dW = \frac{2}{7} m \cdot g \frac{|f'(x)|}{\sqrt{1 + \left(f'(x)\right)^2}} \cdot \sqrt{1 + \left(f'(x)\right)^2} \cdot dx$$

$$= \frac{2}{7} m \cdot g \cdot |f'(x)| \cdot dx$$

Because $dx > 0$, $|dx| = dx$, then:

$$\begin{aligned} dW &= \frac{2}{7} m \cdot g \cdot |f'(x)| \cdot |dx| \\ &= \frac{2}{7} m \cdot g \cdot |f'(x) \cdot dx| \\ &= \frac{2}{7} m \cdot g \cdot |df(x)| \end{aligned}$$

Because the friction force always acts against the movement, the work done by friction is always negative. Then:

$$dW = -\frac{2}{7} m \cdot g \cdot |df(x)|$$

Again taking small displacement Δs along the path:

(16) $$\Delta W = -\frac{2}{7} m \cdot g \cdot \left|\Delta f(x)\right|$$

For the entire path, the total work is the sum of the work done along the little displacements the path had been divided into.

Substituting equation (16) in (13):

$$\begin{aligned}\Delta W &= \Delta K + \Delta U \\ &= K_f - K_i + U_f - U_i \\ &= \tfrac{1}{2} m \cdot v_f^{\,2} - \tfrac{1}{2} m \cdot v_i^{\,2} + m \cdot g \cdot h_f - m \cdot g \cdot h_i\end{aligned}$$

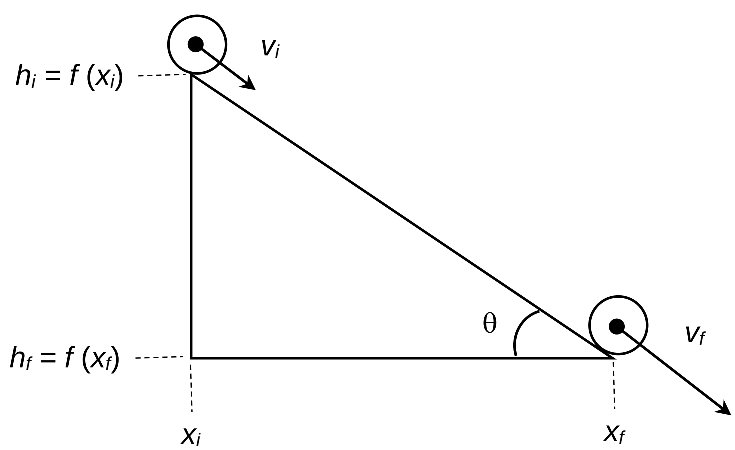
Then: $$-\frac{2}{7} m \cdot g \cdot \left|\Delta f(x)\right| = \tfrac{1}{2} m \cdot v_f^{\,2} - \tfrac{1}{2} m \cdot v_i^{\,2} + m \cdot g \cdot h_f - m \cdot g \cdot h_i$$

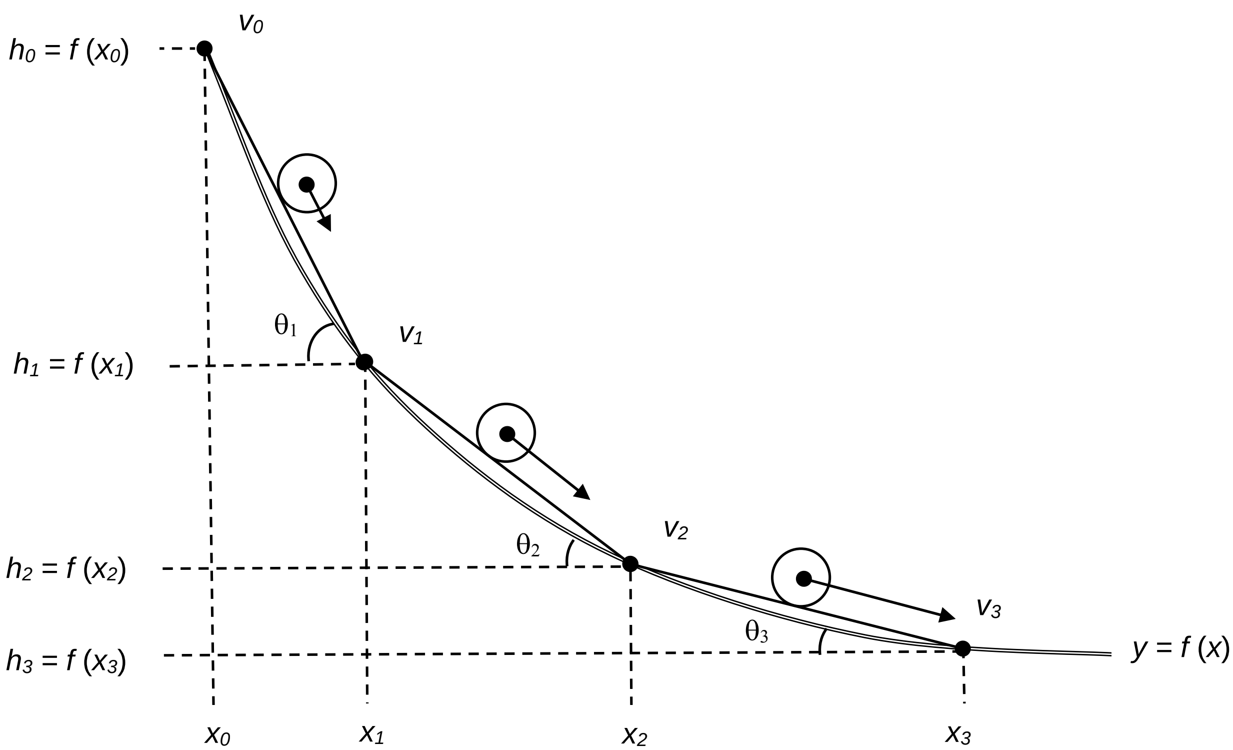
because the height of the object on the path is given by the value of function *f* (*x*) (that is, *h* = *f* (*x*)).

(17) $$-\frac{2}{7} \cdot g \cdot \left| f(x_f) - f(x_i) \right| = \tfrac{1}{2} \cdot v_f^{\,2} - \tfrac{1}{2} \cdot v_i^{\,2} + g \cdot f(x_f) - g \cdot f(x_i)$$

Using the above equation, it is possible to find the velocity of the rolling body at the end of every little displacement, knowing the values of its velocity at the beginning of the displacement, and of the corresponding potential energy change:

(18) $$v_f = \sqrt{v_i^{\,2} - 2g \cdot \left(f(x_f) - f(x_i) \right) - \frac{4}{7} \cdot g \cdot \left| f(x_f) - f(x_i) \right|}$$





A variable slope path can be approximated by a sequence of small inclines, and on each one, the mechanical-energy theorem can be applied, and the work done by the friction force can be easily estimated.