Derivations

This worksheet is intended to document the derivation of the least squares procedure followed to fit Model 1 and Model 2 to acceleration data. It is not needed during the activity, but may be adapted to use in student research or marking period project.

1 Piecewise constant model (Model 1)

We are given acceleration values $a_1, a_2, \cdots, a_N$ for the times $t_1, t_2, \cdots, t_N$. Our goal is to fit a constant c_1 to the data over the interval $0 \leq t \leq t_1$, and another constant c_2 over the interval $t_1 < t \leq t_2$. In doing so, we must make sure that the total area under the fitted graph is equal to zero, corresponding to zero initial and final racer velocities. This is a constrained least squares problem. Let N_1 and N_2 denote the number of acceleration data points that fall in time intervals $0 \leq t \leq t_1$ and $t_1 < t \leq t_2$. We can then minimize

$$F = \sum_{i=1}^{N_1} (c_1 - a_i)^2 + \sum_{i=N_1+1}^{N_2} (c_2 - a_i)^2 \tag{1}$$

subject to the constraint

$$G = c_1 t_1 + c_2 (t_2 - t_1) = 0. \tag{2}$$

We can solve this problem by forming the Lagrangian

$$\mathbf{grad} F = \lambda \mathbf{grad} G, \text{ and } G = 0, \tag{3}$$

which gives us three equations in three unknowns

$$
\begin{aligned}
&2\sum_{i=1}^{N_1}(c_1 - a_i) = \lambda t_1, \\
&2\sum_{i=N_1+1}^{N_2}(c_2 - a_i) = \lambda\left(t_2 - t_1\right), \\
&c_1 t_1 + c_2\left(t_2 - t_1\right) = 0.
\end{aligned}
\tag{4}
$$

Eliminating λ from equations 1 and 2 leaves us with two linear equations in two unknowns c_1 and c_2

$$
\begin{aligned}
&\frac{N_1}{t_1}c_1 + \frac{N_2}{t_1 - t_2}c_2 = \frac{1}{t_1}\sum_{i=1}^{N_1} a_i + \frac{1}{t_1 - t_2}\sum_{i=N_1+1}^{N_2} a_i = \beta \\
&c_1 t_1 + c_2\left(t_2 - t_1\right).
\end{aligned}
\tag{5}
$$

When we solve these two equations for the unknowns c_1 and c_2, we get

$$
\begin{aligned}
c_1 &= \frac{t_1\left(t_1 - t_2\right)^2\beta}{\left(t_1^2 N_2 + \left(t_1 - t_2\right)^2 N_1\right)}, \\
c_2 &= \frac{t_1^2\left(t_1 - t_2\right)\beta}{\left(t_1^2 N_2 + \left(t_1 - t_2\right)^2 N_1\right)},
\end{aligned}
\tag{6}
$$

where β is

$$
\beta = \frac{1}{t_1}\sum_{i=1}^{N_1} a_i + \frac{1}{t_1 - t_2}\sum_{i=N_1+1}^{N_2} a_i. \tag{7}
$$

These formulas are implemented in the Analysis Spreadsheet.

2 Piecewise linear model (Model 2)

We are given acceleration data $a_1, a_2, \cdots, a_N$ along with times $t_1, t_2, \cdots, t_N$, and we want to fit a line $\alpha_1 + \beta_1 t$ to data over the interval $0 \le t \le t_1$ and another line $\alpha_2 + \beta_2 t$ to data over the interval $t_1 < t \le t_2$. Whereas we had to find only two unknowns in the case of the piecewise constant model, in the piecewise linear model, we need to find four unknowns. Let N_1 and

N_2 denote the number of acceleration data points that fall in time intervals $0 \leq t \leq t_1$ and $t_1 < t \leq t_2$. We can then minimize

$$\sum_{i=1}^{N_1} (\alpha_1 + \beta_1 t_i - a_i)^2 + \sum_{i=N_1+1}^{N_2} (\alpha_2 + \beta_2 t_i - a_i)^2 \tag{8}$$

subject to the zero area constraint

$$\alpha_1 t_1 + \frac{1}{2}\beta_1 t_1^2 + \alpha_2 (t_2 - t_1) + \frac{1}{2}\beta_2 \left(t_2^2 - t_1^2\right) = 0. \tag{9}$$

This problem is more easily solved by working with matrices. Let's define a vector of unknowns

$$\theta = \begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \end{pmatrix}^t, \tag{10}$$

two vectors of acceleration measurements

$$\begin{aligned} A_1 &= \begin{pmatrix} a_1 & \cdots & a_{N_1} \end{pmatrix}^t, \\ A_2 &= \begin{pmatrix} a_{N_1+1} & \cdots & a_{N_2} \end{pmatrix}^t, \end{aligned} \tag{11}$$

a vector of constants

$$C = \begin{pmatrix} t_1 & \frac{1}{2}t_1^2 & (t_2 - t_1) & \frac{1}{2}(t_2^2 - t_1^2) \end{pmatrix}^t, \tag{12}$$

and two matrices T_1 and T_2 as

$$T_1 = \begin{pmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_{N_1} \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}^t, \tag{13}$$

$$T_2 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ t_{N_1+1} & \cdots & t_{N_2} \end{pmatrix}^t. \tag{14}$$

We can now write our constrained optimization as follows. Mimimize the quantity

$$(A_1 - T_1\theta)^t (A_1 - T_1\theta) + (A_2 - T_2\theta)^t (A_2 - T_2\theta) \tag{15}$$

with respect to θ, subject to the constraint

$$C^t\theta = 0. \tag{16}$$

To solve this problem, we form the Lagrangian

$$J = \frac{1}{2}\left(A_1 - T_1\theta\right)^t\left(A_1 - T_1\theta\right) + \frac{1}{2}\left(A_2 - T_2\theta\right)^t\left(A_2 - T_2\theta\right) - \left(C^t\theta\right)^t\lambda. \quad (17)$$

Equating the derivative to zero gives us

$$\theta = \left(T_1^tT_1 + T_2^tT_2\right)^{-1}\left(T_1^tA_1 + T_2^tA_2\right) - \left(T_1^tT_1 + T_2^tT_2\right)^{-1}C\lambda. \quad (18)$$

To solve for λ, we can plug in for the constraint to get

$$\lambda = \left(C^t\left(T_1^tT_1 + T_2^tT_2\right)^{-1}C\right)^{-1}C^t\left(\left(T_1^tT_1 + T_2^tT_2\right)^{-1}\left(T_1^tA_1 + T_2^tA_2\right)\right). \quad (19)$$

Substituting this expression for λ in equation (18) gives us the solution

$$\theta = M_1^{-1}M_2 - \frac{M_1^{-1}CC^t}{C^tM_1^{-1}C}M_1^{-1}M_2, \quad (20)$$

where

$$\begin{aligned} M_1 &= T_1^tT_1 + T_2^tT_2 \\ M_2 &= T_1^tA_1 + T_2^tA_2. \end{aligned} \quad (21)$$

This formula is implemented in the Analysis Spreadsheet.